

# Derived categories of toric varieties II

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## Abstract

We prove two theorems on the derived categories of toric varieties, the existence of an exceptional collection consisting of sheaves for a divisorial extraction and the finiteness of Fourier-Mukai partners.

## 1 Introduction

This paper supplements the first part of the series [5], where we considered the semi-orthogonal decompositions of the derived categories with respect to the toric Minimal Model Program. We proved that the semi-orthogonal complements for divisorial contractions and flips have full exceptional collections consisting of sheaves.

Akira Ishii and Kazushi Ueda pointed out that the divisorial extractions, another important class of birational transformations, were not yet treated. Namely we did not consider toric birational morphisms  $f : X \rightarrow Y$  between  $\mathbf{Q}$ -factorial projective toric varieties whose exceptional locus is a prime divisor  $E$  such that  $K_X + eE = f^*K_Y$  with  $e > 0$ . We note that, if  $e < 0$ , then  $f$  is a divisorial contraction which is already treated in [5], and if  $e = 0$ , then  $f$  is a log crepant morphism and we have a derived equivalence ([4]). So we consider this case in §1, and prove that the semi-orthogonal complement has again a full exceptional collection consisting of sheaves. It is rather remarkable because the directions are opposite for the fully faithful embedding functors between derived categories in the cases  $e > 0$  and  $e < 0$ .

We also correct certain notation in [5] §5 (paragraph before Remark 5.1) according to a remark of Akira Ishii and Kazushi Ueda. Namely we write  $j_{1*}j_2^*$  instead of  $j^*$  because there is no morphism of stacks over a morphism of

schemes  $D \rightarrow X$ , but we have a flip-like diagram  $\mathcal{D} \leftarrow \tilde{\mathcal{D}} \rightarrow \mathcal{X}$  (paragraphs between Lemmas 2 and 3).

In §2, we answer a question by Shinnosuke Okawa raised at Chulalongkorn University conference. We prove that the number of Fourier-Mukai partners of a  $\mathbf{Q}$ -factorial projective toric variety is finite, confirming a conjecture which is related to the finiteness conjecture of the minimal models ([2]).

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## 2 Divisorial extraction

We proved in [4] that the Minimal Model Program in the category of toroidal varieties yields semi-orthogonal decompositions of derived categories. In [5] we proved in the toric case that the semi-orthogonal complements have full exceptional collections consisting of sheaves. We extend this result to another case of important toric morphism, a toric divisorial extraction.

**Theorem 1.** *Let  $\phi : X \rightarrow Y$  be a birational morphism between  $\mathbf{Q}$ -factorial projective toric varieties such that the exceptional locus is a prime divisor denoted by  $D$ . Let  $B$  be an effective torus-invariant  $\mathbf{Q}$ -divisor on  $X$  whose coefficients belong to a set*

$$\left\{ \frac{r-1}{r} \mid r \in \mathbf{Z}_{>0} \right\}$$

*and  $C = \phi_* B$ . Let  $\pi_X : \mathcal{X} \rightarrow X$  and  $\pi_Y : \mathcal{Y} \rightarrow Y$  be the natural morphisms from smooth Deligne-Mumford stacks corresponding to the pairs  $(X, B)$  and  $(Y, C)$  respectively. Assume that*

$$K_X + B < \phi^*(K_Y + C).$$

*Then there is a fully faithful triangulated functor*

$$\Phi : D^b(\mathrm{Coh}(\mathcal{X})) \rightarrow D^b(\mathrm{Coh}(\mathcal{Y}))$$

*such that the semi-orthogonal complement of the image  $\Phi(D^b(\mathrm{Coh}(\mathcal{X})))^\perp$  has a full exceptional collection consisting of sheaves.*

*Proof.* This is the case which was not considered in the previous paper [5], where we considered only the case  $K_X + B \geq \phi^*(K_Y + C)$ . The proof is very similar, but we need additional calculations.

We use the notation of [5] whenever it is possible. Especially the morphism  $\phi : X \rightarrow Y$  is controlled by an equation

$$a_1 v_1 + \cdots + a_{n+1} v_{n+1} = 0$$

locally over  $Y$ , where the  $v_i$  are primitive vertices of cones in a decomposition of a cone corresponding to a toric affine open subset  $U$  of  $Y$ , and the coefficients of the equation  $a_i$  are coprime integers. Since  $\phi$  is a divisorial contraction, we have  $a_i \geq 0$  except for  $i = n + 1$ . We assume that  $a_i > 0$  for  $1 \leq i \leq \alpha$ ,  $a_i = 0$  for  $\alpha < i \leq n$ , and  $a_{n+1} < 0$ .

Let  $D_i$  be the prime divisors on  $X$  corresponding to the vertices  $v_i$ , and  $E_i = \phi(D_i)$  for  $1 \leq i \leq n$  their images on  $Y$ .  $D = D_{n+1}$  is the exceptional divisor of  $\phi$ , and the restricted morphism  $\bar{\phi} : D \rightarrow F = \phi(D)$  is a toric Mori fiber space. Let  $\mathcal{D}_i$  and  $\mathcal{E}_i$  respectively be the prime divisors on  $\mathcal{X}$  and  $\mathcal{Y}$  above  $D_i$  and  $E_i$ .

We write  $B|_{\phi^{-1}(U)} = \sum_{i=1}^{n+1} \frac{r_i-1}{r_i} D_i$ , and  $C|_U = \sum_{i=1}^n \frac{r_i-1}{r_i} E_i$ . The assumption that  $K_X + B$  is positive for  $\phi$  is expressed by the following inequality

$$\sum_{i=1}^{n+1} \frac{a_i}{r_i} < 0.$$

The following Lemma 2 is [4] Theorem 4.2 (4):

**Lemma 2.** *Let  $\mathcal{W} = (\mathcal{X} \times_Y \mathcal{Y})^\sim$  be the normalized fiber product, and let  $\mu$  and  $\nu$  be the projections. Then the functor  $\Phi = \nu_* \mu^* : D^b(\text{Coh}(\mathcal{X})) \rightarrow D^b(\text{Coh}(\mathcal{Y}))$  is fully faithful. Moreover, the invertible sheaves  $\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i \mathcal{D}_i)$  for the sequences of integers  $k = (k_1, \dots, k_{n+1})$  such that*

$$0 \leq -\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < \sum_{i=1}^{\alpha} \frac{a_i}{r_i}$$

*span the triangulated category  $D^b(\text{Coh}(\mathcal{X}))$ , and*

$$\Phi(\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i \mathcal{D}_i)) \cong \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n k_i \mathcal{E}_i)$$

*for such sequences of integers.*

*Proof.* We recall the proof for the convenience of the reader. We have

$$\sum_{i=1}^{n+1} k_i \mu^* \mathcal{D}_i \equiv \sum_{i=1}^n k_i \nu^* \mathcal{E}_i + \frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \mu^* \mathcal{D}_{n+1}$$

and

$$K_X + \sum_{i=1}^n \frac{r_i - 1}{r_i} D_i + D_{n+1} = \phi^*(K_Y + \sum_{i=1}^n \frac{r_i - 1}{r_i} E_i) - \sum_{i=1}^n \frac{a_i}{r_i} \frac{1}{a_{n+1}} D_{n+1}.$$

Since

$$\frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < -\frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^n \frac{a_i}{r_i}$$

we have

$$R^q \nu_* \mu^* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i \mathcal{D}_i) \cong 0$$

for  $q > 0$ . Moreover since

$$\frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \geq 0$$

we have

$$\nu_* \mu^* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i \mathcal{D}_i) \cong \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n k_i \mathcal{E}_i).$$

For another such sequence  $k' = (k'_1, \dots, k'_{n+1})$ , we have

$$\frac{a_{n+1}}{r_{n+1}} < -\sum_{i=1}^n \frac{a_i}{r_i} < -\sum_{i=1}^{n+1} \frac{a_i (k_i - k'_i)}{r_i} < \sum_{i=1}^n \frac{a_i}{r_i}$$

hence

$$R^q \nu_* \mu^* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} (k_i - k'_i) \mathcal{D}_i) \cong 0$$

for  $q > 0$  and

$$\nu_* \mu^* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} (k_i - k'_i) \mathcal{D}_i) \cong \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n (k_i - k'_i) \mathcal{E}_i).$$

Thus the natural homomorphism

$$\mathrm{Hom}(L', L) \rightarrow \mathrm{Hom}(\Phi(L'), \Phi(L'))$$

is bijective for  $L = \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i \mathcal{D}_i)$  and  $L' = \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k'_i \mathcal{D}_i)$ .

Since  $\bigcup_{i=1}^{\alpha} \mathcal{D}_i = \emptyset$ , the following Koszul complex is exact:

$$0 \rightarrow \mathcal{O}_{\mathcal{X}}(-\sum_{i=1}^{\alpha} \mathcal{D}_i) \rightarrow \cdots \rightarrow \sum_{i=1}^{\alpha} \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i) \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow 0.$$

Therefore the sheaves  $\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} k_i \mathcal{D}_i)$  whose coefficients  $k_i$  satisfy the assumption of the lemma generate the same category as those whose coefficients are general.  $\square$

We shall construct a full exceptional collection on the semiorthogonal complement of the image of  $\Phi$ .

We define a  $\mathbf{Q}$ -divisor  $\bar{B} = \sum_{i=1}^n \frac{r_i t_i - 1}{r_i t_i} \bar{D}_i$  for  $\bar{D}_i = D_i \cap D$  similarly as in [5] §5. Namely, we write

$$v_i \equiv t_i \bar{v}_i$$

in  $N_D = N_X / \mathbf{Z} v_{n+1}$  for primitive vectors  $\bar{v}_i \in N_D$ , and  $a_i t_i = t \bar{a}_i$  such that the  $\bar{a}_i$  for  $1 \leq i \leq \alpha$  are coprime integers, where  $N_X$  and  $N_D$  are lattices corresponding to the toric varieties  $X$  and  $D$  respectively. Here we note that

$$M_D = \{m \in M_X \mid \langle m, v_{n+1} \rangle = 0\}$$

for the dual lattices  $M_X$  and  $M_D$  of  $N_X$  and  $N_D$  respectively.

The toric morphism  $\bar{\phi} : D \rightarrow F$  is controlled locally over  $F$  by an equation

$$\sum_{i=1}^n \bar{a}_i \bar{v}_i = 0$$

where we have  $\bar{a}_i = 0$  for  $\alpha < i \leq n$ . We have  $D_i|_D = \frac{1}{t_i} \bar{D}_i$  as  $\mathbf{Q}$ -Cartier divisors.

Let  $\pi_D : \mathcal{D} \rightarrow D$  be the natural morphism from a smooth Deligne-Mumford stack corresponding to the pair  $(D, \bar{B})$ . Let  $\tilde{\mathcal{D}} = (\mathcal{X} \times_X D)^\sim$  be the normalized fiber product. Then there are natural morphisms  $j_1 : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$  and  $j_2 : \tilde{\mathcal{D}} \rightarrow \mathcal{X}$  satisfying

$$j_{1*} j_2^* \mathcal{O}_{\mathcal{X}}(\mathcal{D}_i) \cong \mathcal{O}_{\mathcal{D}}(\bar{\mathcal{D}}_i)$$

for  $i = 1, \dots, n$ , where  $\mathcal{D}_i$  and  $\bar{\mathcal{D}}_i$  are prime divisors on  $\mathcal{X}$  and  $\mathcal{D}$  corresponding to  $D_i$  and  $\bar{D}_i$  respectively.

Next we define a  $\mathbf{Q}$ -divisor  $\bar{C}$  on  $F$  by  $\bar{C} = \sum_{i=\alpha+1}^n \frac{r_i s_i t_i - 1}{r_i s_i t_i} \bar{E}_i$  for  $\bar{E}_i = E_i \cap F$  similarly as in [5] §4. Namely, we write

$$\bar{v}_i \equiv s_i \tilde{v}_i$$

for primitive vectors  $\tilde{v}_i$  in the lattice

$$N_F = N_D / \left( \sum_{i=1}^{\alpha} \mathbf{R} v_i \cap N_D \right)$$

corresponding to  $F$ . Here we note that

$$M_F = \{m \in M_F \mid \langle m, v_i \rangle = 0, i = 1, \dots, \alpha\}$$

for the dual lattices  $M_F$  of  $N_F$ .

Let  $\pi_F : \mathcal{F} \rightarrow F$  be the natural morphism from a smooth Deligne-Mumford stack corresponding to the pair  $(F, \bar{C})$ . Let  $\tilde{\mathcal{F}} = (\mathcal{Y} \times_Y F)$  be the normalized fiber product. Then there are natural morphisms  $j_{F,1} : \tilde{\mathcal{F}} \rightarrow \mathcal{F}$  and  $j_{F,2} : \tilde{\mathcal{F}} \rightarrow \mathcal{Y}$  satisfying

$$j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}(\mathcal{E}_i) \cong \mathcal{O}_{\mathcal{F}}(\bar{\mathcal{E}}_i)$$

for  $i = \alpha + 1, \dots, n$ , where  $\mathcal{E}_i$  and  $\bar{\mathcal{E}}_i$  are prime divisors on  $\mathcal{Y}$  and  $\mathcal{F}$  corresponding to  $E_i$  and  $\bar{E}_i$  respectively. Indeed we have an equality  $E_i|_F = \frac{1}{s_i t_i} \bar{E}_i$  for  $\alpha < i \leq n$ , which is confirmed by the following equalities  $\phi^* E_i = D_i$ ,  $D_i|_D = \frac{1}{t_i} \bar{D}_i$  and  $\bar{\phi}^* \bar{E}_i = s_i \bar{D}_i$ .

We have an induced morphism  $\bar{\psi} : \mathcal{D} \rightarrow \mathcal{F}$ . Since  $\bar{\psi}$  is smooth by [5] Corollary 4.2, we have

$$\bar{\psi}^* \mathcal{O}_{\mathcal{F}}(\bar{\mathcal{E}}_i) \cong \mathcal{O}_{\mathcal{D}}(\bar{\mathcal{D}}_i)$$

for  $\alpha < i \leq n$ .

**Lemma 3.** *Let  $k_1, \dots, k_n$  be integers, and define  $k_{n+1}$  by an equation*

$$\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = 0.$$

*If  $k_{n+1}$  is not an integer which is divisible by  $r_{n+1}$ , then*

$$j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}} \left( \sum_{i=1}^{\alpha} k_i \mathcal{E}_i \right) \cong 0.$$

*Proof.* Since  $j_{F,2}^* \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i)$  is an invertible sheaf, its direct image sheaf is either an invertible sheaf or a zero sheaf on  $\mathcal{F}$ . If it is an invertible sheaf, then its pull-back

$$\bar{\psi}^* j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i)$$

is also an invertible sheaf, which should be of the form

$$j_{1*} j_2^* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{\alpha} k_i \mathcal{D}_i + k_{n+1} \mathcal{D}_{n+1})$$

for an integer  $k_{n+1}$  such that  $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = 0$ . Since it is non-zero if and only if  $k_{n+1}$  is divisible by  $r_{n+1}$ , we conclude our proof.  $\square$

Our theorem is a consequence of the following Proposition 4 combined with [5] Theorem 1.1.

**Proposition 4.** (1) *The functor*

$$j_{F,2*} j_{F,1}^* : D^b(\text{Coh}(\mathcal{F})) \rightarrow D^b(\text{Coh}(\mathcal{Y}))$$

*is fully faithful.*

*Let  $D^b(\text{Coh}(\mathcal{F}))_k$  denote the full subcategory of  $D^b(\text{Coh}(\mathcal{Y}))$  defined by*

$$D^b(\text{Coh}(\mathcal{F}))_k = j_{F,2*} j_{F,1}^* D^b(\text{Coh}(\mathcal{F})) \otimes \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i)$$

*for a sequence of integers  $k = (k_1, \dots, k_{\alpha})$ . We set  $k_{\alpha+1} = \dots = k_n = 0$  when necessary.*

(2) *If*

$$0 < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \leq - \sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

*for some integer  $k_{n+1}$ , then*

$$\text{Hom}^q(\Phi(D^b(\text{Coh}(\mathcal{X}))), D^b(\text{Coh}(\mathcal{F}))_k) = 0$$

*for all  $q$ .*

(3) Let  $k' = (k'_1, \dots, k'_\alpha)$  be another sequence of integers such that

$$0 < \sum_{i=1}^{n+1} \frac{a_i(k'_i - k_i)}{r_i} < -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

for some integers  $k_{n+1}$  and  $k'_{n+1}$ . Then

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{F}))_k, D^b(\mathrm{Coh}(\mathcal{F}))_{k'}) = 0$$

for all  $q$ .

(4) The subcategories  $\Phi(D^b(\mathrm{Coh}(\mathcal{X})))$  and the  $D^b(\mathrm{Coh}(\mathcal{F}))_k$  for

$$0 < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \leq -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

for some integers  $k_{n+1}$  generate  $D^b(\mathrm{Coh}(\mathcal{Y}))$  as a triangulated category.

*Proof.* (1) We shall prove that the natural homomorphisms

$$\mathrm{Hom}^q(L, L') \rightarrow \mathrm{Hom}^q(j_{F,2*}j_{F,1}^*L, j_{F,2*}j_{F,1}^*L') \cong \mathrm{Hom}^q(j_{F,1*}j_{F,2}^*j_{F,2*}j_{F,1}^*L, L')$$

are bijective for all  $q$  and for all invertible sheaves  $L$  and  $L'$  on  $\mathcal{F}$ .

Let  $V = \sum_{i=1}^{\alpha} \mathcal{O}_{\mathcal{Y}}(-\mathcal{E}_i)$ . Then there is a Koszul resolution of  $j_{F,2*}j_{F,1}^*\mathcal{O}_{\mathcal{F}}$ :

$$0 \rightarrow \bigwedge^{\alpha} V \rightarrow \dots \rightarrow V \rightarrow \mathcal{O}_{\mathcal{Y}} \rightarrow j_{F,2*}j_{F,1}^*\mathcal{O}_{\mathcal{F}} \rightarrow 0.$$

Thus

$$H_q(j_{F,1*}j_{F,2}^*j_{F,2*}j_{F,1}^*\mathcal{O}_{\mathcal{F}}) \cong \bigoplus_{\#I=q} j_{F,1*}j_{F,2}^*\mathcal{O}_{\mathcal{Y}}(-\sum_{i \in I} \mathcal{E}_i)$$

where the  $I$  run all the subsets of  $\{1, \dots, \alpha\}$  such that  $\#I = q$ . Since  $\sum_{i=1}^{n+1} \frac{a_i}{r_i} < 0$ , we have

$$0 < \sum_{i \in I} \frac{a_i}{r_i} < \frac{|a_{n+1}|}{r_{n+1}}$$

for  $q \neq 0$ . Therefore we have

$$H_q(j_{F,1*}j_{F,2}^*j_{F,2*}j_{F,1}^*\mathcal{O}_{\mathcal{F}}) \cong 0$$



for such  $q$  by Lemma 3. By the projection formula, we conclude that

$$j_{F,1*}j_{F,2}^*j_{F,2*}j_{F,1}^*L \cong L$$

hence the assertion.

(2) By Lemma 2 or [4] Theorem 4.2 (4),  $\Phi(D^b(\text{Coh}(\mathcal{X}))$  is spanned by invertible sheaves  $\mathcal{O}_Y(\sum_{i=1}^n l_i \mathcal{E}_i)$  for

$$0 \leq -\sum_{i=1}^{n+1} \frac{a_i l_i}{r_i} < \sum_{i=1}^{\alpha} \frac{a_i}{r_i}$$

where  $l_{n+1}$  are some integers. By adding the inequalities, we obtain

$$0 < -\sum_{i=1}^{n+1} \frac{a_i(l_i - k_i)}{r_i} < -\frac{a_{n+1}}{r_{n+1}}.$$

Therefore for any invertible sheaf  $L$  on  $\mathcal{F}$ , we have

$$\begin{aligned} & \text{Hom}^q(\mathcal{O}_Y(\sum_{i=1}^n l_i \mathcal{E}_i), j_{F,2*}j_{F,1}^*L \otimes \mathcal{O}_Y(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i)) \\ & \cong \text{Hom}^q(j_{F,1*}j_{F,2}^*\mathcal{O}_Y(\sum_{i=1}^n (l_i - k_i) \mathcal{E}_i), L) \cong 0 \end{aligned}$$

for all  $q$  by Lemma 3.

(3) For any fixed subset  $I \subset \{1, \dots, \alpha\}$ , we set  $\epsilon_i = 1$  or  $0$  according to whether  $i \in I$  or not. Then we have

$$0 > \sum_{i=1}^{n+1} \frac{a_i(k_i - k'_i - \epsilon_i)}{r_i} > \frac{a_{n+1}}{r_{n+1}}.$$

Therefore for any invertible sheaves  $L$  and  $L'$  on  $\mathcal{F}$ , we have

$$\begin{aligned} & \text{Hom}^q(j_{F,2*}j_{F,1}^*L \otimes \mathcal{O}_Y(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i), j_{F,2*}j_{F,1}^*L' \otimes \mathcal{O}_Y(\sum_{i=1}^{\alpha} k'_i \mathcal{E}_i)) \\ & \text{Hom}^q(j_{F,1*}j_{F,2}^*j_{F,2*}j_{F,1}^*L \otimes \mathcal{O}_Y(\sum_{i=1}^{\alpha} (k_i - k'_i) \mathcal{E}_i), L') \cong 0 \end{aligned}$$

for all  $q$  by Lemma 3.

(4) Let  $T$  be the triangulated subcategory of  $D^b(\text{Coh}(\mathcal{Y}))$  generated by  $\Phi(D^b(\text{Coh}(\mathcal{X})))$  and the  $D^b(\text{Coh}(\mathcal{F}))_k$  for

$$0 < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \leq - \sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

for some integers  $k_{n+1}$ . We shall prove that  $T$  contains all invertible sheaves of the form

$$\mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n k_i \mathcal{E}_i).$$

By Lemma 2,  $T$  contains invertible sheaves  $\mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n k_i \mathcal{E}_i)$  if

$$- \sum_{i=1}^{\alpha} \frac{a_i}{r_i} < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \leq 0$$

for some integers  $k_{n+1}$ . On the other hand, for an arbitrary sequence of integers  $(k_1, \dots, k_n)$ , there exists an integer  $k_{n+1}$  such that

$$- \sum_{i=1}^{\alpha} \frac{a_i}{r_i} < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \leq - \sum_{i=1}^{n+1} \frac{a_i}{r_i}.$$

The difference of the above two intervals is covered by the consideration on the Koszul resolution of the sheaf  $j_{F,2*} j_{F,1}^* \mathcal{O}_{\mathcal{F}}$ . Indeed assume that  $k = (k_1, \dots, k_{n+1})$  is a sequence of integers such that

$$0 < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \leq - \sum_{i=1}^{n+1} \frac{a_i}{r_i}.$$

If  $T$  contains all the invertible sheaves of the form

$$\mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n l_i \mathcal{E}_i)$$

such that  $l_i = k_i$  or  $k_i - 1$ ,  $l_{n+1} = k_{n+1}$  and that  $\sum_{i=1}^n l_i < \sum_{i=1}^n k_i$ , then  $T$  also contains  $\mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n k_i \mathcal{E}_i)$ , because  $D^b(\text{Coh}(\mathcal{F}))_k$  is contained in  $T$ . Therefore by the induction on  $\sum_{i=1}^{n+1} k_i$ , we obtain the assertion.  $\square$

If  $k = (k_1, \dots, k_{n+1})$  and  $k' = (k'_1, \dots, k'_{n+1})$  are sequences of integers such that

$$\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = \sum_{i=1}^{n+1} \frac{a_i k'_i}{r_i}$$

then  $j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^{\alpha} (k_i - k'_i) \mathcal{E}_i)$  is either an invertible sheaf on  $\mathcal{F}$  or 0. In the former case we have  $D^b(\text{Coh}(\mathcal{F}))_k = D^b(\text{Coh}(\mathcal{F}))_{k'}$ , while in the latter case we have

$$\text{Hom}^q(D^b(\text{Coh}(\mathcal{F}))_k, D^b(\text{Coh}(\mathcal{F}))_{k'}) = 0$$

for all  $q$ . Therefore we obtain an exceptional collection of the semiorthogonal complement.  $\square$

### 3 Fourier-Mukai partners

In the minimal model program, we conjectured that, given a smooth projective variety, or more generally a smooth projective pair, there exist only finitely many minimal models, or log minimal models, up to isomorphisms which are birationally equivalent to the given variety. In the derived setting, we conjecture that, given a smooth projective variety, there exist only finitely many smooth projective varieties up to isomorphisms whose derived categories are equivalent to the given one ([2]). We also expect more generalized statement to hold for projective varieties with only quotient singularities. We add here one more example which confirms this conjecture.

**Theorem 5.** *Let  $X$  be a projective  $\mathbf{Q}$ -factorial toric variety and  $Y$  a projective variety which has only quotient singularities. Let  $\mathcal{X}$  and  $\mathcal{Y}$  be smooth Deligne-Mumford stacks associated to  $X$  and  $Y$  respectively. Assume that there exists an equivalence of triangulated categories  $\Phi : D^b(\mathcal{X}) \cong D^b(\mathcal{Y})$ . Then  $Y$  is also a projective  $\mathbf{Q}$ -factorial toric variety, and the kernel object for  $\Phi$  induces a toric birational map  $\phi : X \dashrightarrow Y$ . In particular,  $Y$  has only abelian quotient singularities. Moreover there exist only finitely many such birational maps when  $X$  is fixed and  $Y$  is varied.*

*Proof.* Let  $E \in D^b(\mathcal{X} \times \mathcal{Y})$  be the kernel of  $\Phi$ , an object giving the equivalence  $\Phi$  ([3]). Then we have an isomorphism

$$E \otimes p_1^* \omega_{\mathcal{X}} \cong E \otimes p_2^* \omega_{\mathcal{Y}}$$

where  $p_1$  and  $p_2$  are projections.

Let  $T$  be the torus contained in  $X$  and  $B = X \setminus T$ . We take a general point  $y \in \mathcal{Y}$  such that the support of  $E_y = \Psi(\mathcal{O}_y)$  contains a point in  $T$ , where  $\Psi : D^b(\mathcal{Y}) \cong D^b(\mathcal{X})$  is the equivalence given by  $\Psi(a) = p_{1*}(p_2^*a \otimes E)$ . Since  $K_X + B \sim 0$ , we have an isomorphism

$$E_y \cong E_y \otimes \mathcal{O}_X(-B).$$

It follows that the support of  $E_y$  is a point, say  $x \in T$ , by the same argument as in the proof of [2] Theorem 2.3 (2).

If we take another such point  $y'$ , then  $E_{y'}$  is supported by a different point  $x' \in T$ , because we have  $\text{Hom}(\mathcal{O}_y, \mathcal{O}_{y'}[p]) = 0$  for all  $p$ . Therefore the support of  $E$  gives a birational map  $\phi : X \dashrightarrow Y$ . Let  $Z \subset X \times Y$  be the graph of  $\phi$ . By the first isomorphism, we have  $q_1^*K_X = q_2^*K_Y$ , where  $q_1 : Z \rightarrow X$  and  $q_2 : Z \rightarrow Y$  are projections, i.e.,  $X$  and  $Y$  are  $K$ -equivalent.

Let us consider the case where  $\phi$  is not surjective in codimension 1. Let  $D_Y$  be a prime divisor on  $Y$  whose center on  $X$  is not a divisor. Since  $X$  and  $Y$  are  $K$ -equivalent, there exists a crepant divisorial extraction  $\alpha : X' \rightarrow X$  whose exceptional divisor is the prime divisor  $D_X$  corresponding to  $D_Y$ . Indeed there exists a toric projective birational morphism  $\beta : X'' \rightarrow X$  such that all divisorial valuations whose log discrepancies are at most 1 appear as prime divisors on  $X''$ . We note that  $\beta$  is toric because it is obtained by blowing up a relative minimal model of  $X$ , which is toric, at centers which are also toric. Then we can construct  $\alpha$  by contracting all exceptional divisors except  $D_X$ .

Since  $\alpha$  is a crepant toric morphism, we have an equivalence  $D^b(\mathcal{X}) \cong D^b(\mathcal{X}')$  for the smooth Deligne-Mumford stack  $\mathcal{X}'$  associated to  $X'$  ([4]). By replacing  $X$  by  $X'$  if necessary, we may assume that  $\phi$  is surjective in codimension 1.

By the paragraph after [1] Corollary 2.4,  $X$  is a Mori dream space. By Definition 1.10 of loc. cit., all birational maps  $\phi$  from  $X$  which are surjective in codimension 1 are obtained in the following way: there exist finitely many birational maps  $f_i : X \dashrightarrow X_i$  corresponding to the chambers of the movable cone  $\text{Mov}(X)$ , and  $\phi$  coincides with the composition of  $f_i$  and a morphism from  $X_i$  corresponding to one of the finitely many faces of the nef cone  $\text{Nef}(X_i)$ . Moreover all of them are toric maps and morphisms to toric varieties. Therefore we proved the finiteness of the birational maps  $\phi$ .  $\square$

**Remark 6.** More generally, the same statement with the same proof holds if

$K_X$  or  $-K_X$  supports a big divisor and  $X$  as well as its crepant blowing-ups are Mori dream spaces .

## References

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